BRED VECTORS, LYAPUNOV VECTORS, AND DATA ASSIMILATION

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Regional loss of predictability is usually an indication of the presence of a regional instability of the underlying flow, where small errors in the initial conditions (or imperfections in the model) grow to large amplitudes in finite times. The stability properties of evolving flows have been studied using Lyapunov vectors (e.g., Alligood et al, 1996, Ott, 1993, Kalnay, 2002), singular vectors (e.g., Lorenz, 1965, Farrell, 1988, Molteni and Palmer, 1993), and, more recently, with bred vectors (e.g., Szunyogh et al, 1997, Cai et al, 2002). Bred vectors (BVs) are, by construction, closely related to Lyapunov vectors (LVs). In fact, after an infinitely long breeding time, and with the use of infinitesimal amplitudes, bred vectors are identical to leading Lyapunov vectors. In practical applications, however, bred vectors are different from Lyapunov vectors in two important ways: a) bred vectors are never globally orthogonalized and are intrinsically *local in space and time*, and b) they are *finite-amplitude, finite-time vectors*. These two differences are very significant in a dynamical system like the atmosphere whose size is very large. For example, there is "room" in the atmosphere for several synoptic scale instabilities (e.g., storms) to develop independently in different regions (say, North America and Australia), and there are several different possible types of instabilities (such as barotropic, baroclinic, convective, and even Brownian motion), each of them characterized by finite lifetimes. For such a large, complex system, the notion of waiting an infinite time to obtain a single globally dominant Lyapunov vector does not seem to make physical sense.

Bred vectors share some of their properties with leading LVs (Corazza et al, 2001a, 2001b, Toth and Kalnay, 1993, 1997, Cai et al, 2001): 1) Bred vectors are independent of the norm used to define the size of the perturbation. Corazza et al. (2001) showed that bred vectors obtained using a potential enstrophy norm were indistinguishable from bred vectors obtained using a streamfunction squared norm, in striking contrast with singular vectors. 2) Bred vectors are independent of the length of the rescaling period as long as the perturbations remain approximately linear (for example, for atmospheric models the interval for rescaling could be varied between a single time step and 1-2 days without affecting qualitatively the characteristics of the bred vectors).

However, the finite-amplitude, finite-time, and lack of orthogonalization properties of the BVs introduce important differences with LVs:

1) In regions that undergo strong instabilities, the bred vectors tend to be locally dominated by simple, low-dimensional structures. Patil et al (2001) showed that the BV-dim, $\Psi(\sigma_1,\sigma_2,...,\sigma_k)=\left(\sum_{i=1}^{k}\sigma_i\right)^2\Big/\sum_{i=1}^{k}\sigma_i^2$ where σ_i are the square roots of the eigenvalues of the local BV covariance matrix, gives a good estimate of the number of dominant directions (shapes) of the local k bred vectors. They showed that the regions with low dimensionality cover about 20% of the atmosphere. They also found that these low-dimensionality regions have a very well defined vertical structure, and a typical lifetime of 3-7 days. The low dimensionality identifies regions where the instability of the basic flow has manifested itself in a low number of preferred directions of perturbation growth.

2) Using a Quasi-Geostrophic simulation system of data assimilation developed by Morss (1999), Corazza et al (2001a, b) found that bred vectors have structures that *closely resemble the errors of the short forecasts used as first guess* (background errors), which in turn dominate the local analysis errors. This is especially true in regions of low dimensionality, which is not surprising if these are unstable regions where errors grow in preferred shapes.

3) The number of bred vectors needed to represent the unstable subspace in the QG system is small (about 6-10). This was shown by computing the local BV-dim as a function of the number of independent bred vectors. Convergence in the local dimension starts to occur at about 6 BVs, and is essentially complete when the number of vectors is about 10-15 (Corazza et al, 2001a). This should be contrasted with the results of Snyder and Joly (1998) and Palmer et al (1998) who showed that hundreds of Lyapunov vectors with positive Lyapunov exponents are needed to represent the attractor of the system in quasi-geostrophic models.

4) Since only a few bred vectors are needed, and background errors project strongly in the subspace of bred vectors, Corazza et al (2001b) were able to develop cost-efficient methods to improve the 3D-Var data assimilation by adding to the background error covariance terms proportional to the outer

product of the bred vectors, thus representing the "errors of the day". This approach led to a reduction of analysis error variance of about 40% at very low cost.

5) The finite amplitude of the BVs provides a natural filter of fast but irrelevant instabilities due to nonlinear stauration. As shown by Lorenz (1996) Lyapunov vectors (and singular vectors) of models including these physical phenomena would be dominated by the fast but small amplitude instabilities, unless they are explicitly excluded from the linearized models.

6) Every bred vector is qualitatively similar to the locally *leading* LV. LVs beyond the leading LV are obtained by orthogonalization after each time step with respect to the previous LVs subspace. The orthogonalization requires the introduction of a norm. Using an enstrophy norm, the successive LVs have larger and larger horizontal scales, and a choice of a stream function norm would lead to successively smaller scales in the LVs. Beyond the first few LVs, there is little qualitative similarity between the background errors and the LVs.

In summary, in a system like the atmosphere with enough physical space for several independent local instabilities, BVs and LVs share some properties but they also have significant differences. BV are finite-amplitude, finite-time, and because they are not globally orthogonalized, they have local properties in space. Bred vectors are akin to the leading LV, but bred vectors derived from different arbitrary initial perturbations remain distinct from each other, instead of collapsing into a single leading vector, presumably because the nonlinear terms and physical parameterizations introduce sufficient stochastic forcing to avoid such convergence. As a result, there is no need for global orthogonalization, and the number of bred vectors required to describe the natural instabilities in an atmospheric system (from a local point of view) is much smaller than the number of Lyapunov vectors with positive Lyapunov exponents. The BVs are independent of the norm, whereas the LVs beyond the first one do depend on the choice of norm: for example, they become larger in scale with a vorticity norm, and smaller with a stream function norm.

These properties of BVs result in significant advantages for data assimilation and ensemble forecasting for the atmosphere. Errors in the analysis have structures very similar to bred vectors, and it is found that they project very strongly on the subspace of a few bred vectors. This is not true for either Lyapunov vectors beyond the leading LVs, or for singular vectors unless they are constructed with a norm based on the analysis error covariance matrix (or a bred vector covariance). The similarity between bred vectors and analysis errors leads to the ability to include "errors of the day" in the background error covariance and a significant improvement of the analysis beyond 3D-Var at a very low cost (Corazza et al, 2001b).

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